ON A FINITE-TIME STABILITY CRITERION FOR DISCRETE-TIME DELAY NEURAL NETWORKS WITH SECTOR-BOUNDED NEURON ACTIVATION FUNCTIONS

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ABSTRACT

This paper investigates the problem of finite-time stability for discrete-time neural networks with sector-bounded neuron activation functions and interval-like time-varying delay. The extended reciprocally convex approach is used to establish a delay-dependent sufficient condition to ensure finite-time stability for this class of systems. A numerical example to illustrate the effectiveness of the proposed criterion is also included.

Key words: Discrete-time neural networks, finite-time stability, linear matrix inequalities, time-varying delay.

1. INTRODUCTION

In recent decades, neural networks (NNs) with delays have received considerable attention in analysis and synthesis because their wide applications have been realized in various fields, such as image processing, signal processing, pattern recognition, association memory, etc. [1].

The study of dynamic properties of systems over a finite interval of time comes from many reality systems, such as biochemical reaction systems, communication network systems, etc. [2]. For the class of discrete-time NNs, there have been some papers dealing with finite-time stability and boundedness [3, 4]. On the other hand, from [5], we know that nonlinear functions satisfying the sector-bounded condition are more general than the usual class of Lipschitz functions. However, up to this point, only a few authors have investigated general NNs with activation functions satisfying the sector-bounded condition [6, 7]. That motivated our current study. More specifically, in this paper, we suggest conditions that guarantee the finite-time stability of discrete-time delay NNs with sector-bounded neuron activation functions.

The outline of the paper is as follows. Section 2 presents the definition of finite-time stability and some technical propositions necessary for the proof of the main result. A delay-dependent criterion in the form of matrix inequalities for finite-time stability and an illustrative example is presented in Section 3. The paper ends with conclusions and cited references.

Notation: $\mathbb{Z}_+$ denotes the set of all non-negative integers; $\mathbb{R}^n$ denotes the n-dimensional space with the scalar product $x^\mathrm{T}y$; $\mathbb{R}^{n\times r}$ denotes the space of $(n\times r)-$dimension matrices; A^T denotes the transpose of matrix A; A is positive definite $(A>0)$ if $x^\mathrm{T}Ax>0$ for all $x\neq 0$; $A>B$ means $A-B>0$. The notation diag$\{\ldots\}$ stands for a block-diagonal matrix. The symmetric term in a matrix is denoted by $*$.

2. PRELIMINARIES

Consider the following discrete-time neural networks with time-varying delays

$$\begin{cases} x(k+1)=Ax(k)+Wf(x(k))+W_1g(x(k-h(k))),\ k\in\mathbb{Z}_+, \\ x(k)=\varphi(k),\ k\in\{-h_2,-h_2+1,\ldots,0\}, \end{cases} \tag{1}$$

where $x(k)\in\mathbb{R}^n$ is the state vector; n is the number of neurons; the diagonal matrix A represents the self-feedback terms; the matrices $W,W_1\in\mathbb{R}^{n\times n}$ are connection weight matrices. $f(x(k))$ and $g(x(k-h(k)))$ are the neuron activation functions. The time-varying delay function $h(k)$ satisfies the condition

$$0<h_1\leqslant h(k)\leqslant h_2\quad \forall k\in\mathbb{Z}_+, \tag{2}$$

where h_1,h_2 are given positive integers; $\varphi(k)$ is the initial function.

In this paper, we use the following assumption for the neuron activation functions.

Assumption 2.1. [6] The neuron state-based nonlinear functions

$$f(x(k))=[f_1(x_1(k))\quad f_2(x_2(k))\quad \ldots\quad f_n(x_n(k))]^\mathrm{T},$$

$$g(x(k-h(k)))=[g_1(x_1(k-h(k)))\quad g_2(x_2(k-h(k)))\quad \ldots\quad g_n(x_n(k-h(k)))]^\mathrm{T}$$

are continuous and satisfy $f_i(0)=0, g_i(0)=0$ for $i=1,\ldots,n$ and the following sector-bounded conditions

$$\begin{aligned} &[f(x)-f(y)-U_1(x-y)]^\mathrm{T}\quad [f(x)-f(y)-U_2(x-y)]\leqslant 0,\\ &[g(x)-g(y)-V_1(x-y)]^\mathrm{T}\quad [g(x)-g(y)-V_2(x-y)]\leqslant 0, \end{aligned} \tag{3}$$

where U_1,U_2,V_1 and V_2 are real matrices of appropriate dimensions.

Remark 2.1. Observe that, when $U_1=-U_2=U$ and $V_1=-V_2=V$, condition (3) becomes

$$[f(x)-f(y)]^\mathrm{T}[f(x)-f(y)]\leqslant [x-y]^\mathrm{T}U^\mathrm{T}U[x-y],$$

$$[g(x)-g(y)]^{\mathrm{T}}[g(x)-g(y)] \leqslant [x-y]^{\mathrm{T}}V^{\mathrm{T}}V[x-y].$$

This implies that the standard Lipschitz conditions $\| f(x)-f(y) \| \leqslant \| U(x-y) \|$ and $\| g(x)-g(y) \| \leqslant \| V(x-y) \|$ will be satisfied. Therefore, under Assumption 2.1, the neural network model (1) is more general than those considered in [1, 4].

Definition 2.1. (Finite-time stability) Given positive constants c_1, c_2, N with $c_1 < c_2, N \in \mathbb{Z}_+$ and a symmetric positive-definite matrix R, the system (1) is said to be finite-time stable w.r.t. (c_1, c_2, R, N) if

$$\max_{k\in\{-h_2,-h_2+1,\dots,0\}} \varphi^{\mathrm{T}}(k)R\varphi(k) \leqslant c_1 \implies x^{\mathrm{T}}(k)Rx(k) < c_2 \quad \forall k \in \{1,2,\dots,N\}.$$

What follows are some technical propositions that will be used to prove the main result.

Proposition 2.1. (Discrete Jensen Inequality [8]). *For any matrix $M \in \mathbb{R}^{n\times n}, M = M^{\mathrm{T}} > 0$, positive integers r_1, r_2 satisfying $r_1 \leqslant r_2$, a vector function $\omega: \{r_1, r_1+1, \dots, r_2\} \to \mathbb{R}^n$, then*

$$\left(\sum_{i=r_1}^{r_2} \omega(i)\right)^{\mathrm{T}} M \left(\sum_{i=r_1}^{r_2} \omega(i)\right) \leqslant (r_2 - r_1 + 1)\sum_{i=r_1}^{r_2} \omega^{\mathrm{T}}(i)M\omega(i).$$

Proposition 2.2. (Extended Reciprocally Convex Matrix Inequality [9]). *Let $R \in \mathbb{R}^{n\times n}$ be a symmetric positive-definite matrix. Then the following matrix inequality*

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ 0 & \frac{1}{1-\alpha}R \end{bmatrix} \geqslant \begin{bmatrix} R+(1-\alpha)T_1 & S \\ * & R+\alpha T_2 \end{bmatrix},$$

holds for some matrix $S \in \mathbb{R}^{n\times n}$ and for all $\alpha \in (0,1)$, where $T_1 = R - SR^{-1}S^{\mathrm{T}}$, $T_2 = R - S^{\mathrm{T}}R^{-1}S$.

Proposition 2.3. (Schur Complement Lemma [10]). *Given constant matrices X, Y, Z with appropriate dimensions satisfying $X = X^{\mathrm{T}}, Y = Y^{\mathrm{T}} > 0$. Then*

$$X + Z^{\mathrm{T}}Y^{-1}Z < 0 \iff \begin{bmatrix} X & Z^{\mathrm{T}} \\ Z & -Y \end{bmatrix} < 0.$$

3. MAIN RESULT

Let $h_{12} = h_2 - h_1$, $y(k) = x(k+1) - x(k)$ and let τ be some positive real constant such that the following estimate

$$\max_{k\in\{-h_2,-h_2+1,\dots,-1\}} y^{\mathrm{T}}(k)y(k) < \tau$$

holds. We define the following matrices to facilitate the presentation of the main result.

$\overline{U}_1 = \frac{1}{2}\left(U_1^{\mathrm{T}}U_2 + U_2^{\mathrm{T}}U_1\right)$, $\overline{U}_2 = -\frac{1}{2}\left(U_1^{\mathrm{T}} + U_2^{\mathrm{T}}\right)$,

$\overline{V}_1 = \frac{1}{2}\left(V_1^{\mathrm{T}}V_2 + V_2^{\mathrm{T}}V_1\right)$, $\overline{V}_2 = -\frac{1}{2}\left(V_1^{\mathrm{T}} + V_2^{\mathrm{T}}\right)$,

$\Omega^{11} = -\delta(P + S_1) + (h_{12} + 1)Q + R_1 - \overline{U}_1,\ \Omega^{12} = \delta S_1,\ \Omega^{15} = -\overline{U}_2,$

$\Omega^{17} = AP,\ \Omega^{18} = h_1^2(A - I)S_1,\ \Omega^{19} = h_{12}^2(A - I)S_2,$

$\Omega_{h_1}^{22} = \delta^{h_1}(-R_1 + R_2 - 2\delta S_2) - \delta S_1,\ \Omega_{h_2}^{22} = \delta^{h_1}(-R_1 + R_2 - \delta S_2) - \delta S_1,$

$\Omega_{h_1}^{23} = \delta^{h_1+1}(2S_2 - S),\ \Omega_{h_2}^{23} = \delta^{h_1+1}(S_2 - S),\ \Omega^{24} = \delta^{h_1+1}S,\ \Omega_{h_1}^{2,10} = -\delta^{h_1+1}S,$

$\Omega^{33} = -\delta^{h_1}Q - \delta^{h_1+1}(3S_2 - S - S^{\mathrm{T}}) - \overline{V}_1,\ \Omega_{h_1}^{34} = \delta^{h_1+1}(S_2 - S),$

$\Omega_{h_2}^{34} = \delta^{h_1+1}(2S_2 - S),\ \Omega^{36} = -\overline{V}_2,\ \Omega_{h_1}^{3,10} = \delta^{h_1+1}S,\ \Omega_{h_2}^{3,10} = -\delta^{h_1+1}S^{\mathrm{T}},$

$\Omega_{h_1}^{44} = -\delta^{h_2}R_2 - \delta^{h_1+1}S_2,\ \Omega_{h_2}^{44} = -\delta^{h_2}R_2 - 2\delta^{h_1+1}S_2,\ \Omega_{h_2}^{4,10} = \delta^{h_1+1}S^{\mathrm{T}},$

$\Omega^{55} = \Omega^{66} = -I,\ \Omega^{57} = W^{\mathrm{T}}P,\ \Omega^{58} = h_1^2W^{\mathrm{T}}S_1,\ \Omega^{59} = h_{12}^2W^{\mathrm{T}}S_2,$

$\Omega^{67} = W_1^{\mathrm{T}}P,\ \Omega^{68} = h_1^2W_1^{\mathrm{T}}S_1,\ \Omega^{69} = h_{12}^2W_1^{\mathrm{T}}S_2,$

$\Omega^{77} = -P,\ \Omega^{88} = -h_1^2S_1,\ \Omega^{99} = -h_{12}^2S_2,\ \Omega^{10,10} = -\delta^{h_1+1}S_2,$

$\rho_1 = \frac{1}{2}c_1(h_1 + h_2)(h_{12} + 1)\delta^{N+h_2},\ \rho_2 = \frac{1}{2}\tau h_{12}^2(h_1 + h_2 + 1)\delta^{N+h_2},$

$\Lambda^{11} = -c_2\delta\lambda_1,\ \Lambda^{12} = c_1\delta^{N+1}\lambda_2,\ \Lambda^{13} = \rho_1\lambda_3,\ \Lambda^{14} = c_1h_1\delta^{N+h_1}\lambda_4,$

$\Lambda^{15} = c_1h_{12}\delta^{N+h_2}\lambda_5,\ \Lambda^{16} = \frac{1}{2}\tau h_1^2(h_1 + 1)\delta^{N+h_1}\lambda_6,\ \Lambda^{17} = \rho_2\lambda_7,$

$\Lambda^{22} = -c_1\delta^{N+1}\lambda_2,\ \Lambda^{33} = -\rho_1\lambda_3,\ \Lambda^{44} = -c_1h_1\delta^{N+h_1}\lambda_4,$

$\Lambda^{55} = -c_1h_{12}\delta^{N+h_2}\lambda_5,\ \Lambda^{66} = -\frac{1}{2}\tau h_1^2(h_1 + 1)\delta^{N+h_1}\lambda_6,\ \Lambda^{77} = -\rho_2\lambda_7,$

$\Lambda^{ij} = 0$ for any other $i,j\colon\ j > i,\ \Lambda^{ij} = \left(\Lambda^{ji}\right)^{\mathrm{T}}, i > j.$

Theorem 3.1. *Given positive constants c_1, c_2, γ, N with $c_1 < c_2, N \in \mathbb{Z}_+$ and a symmetric positive-definite matrix R. System (1) is finite-time stable w.r.t. (c_1, c_2, R, N) if there exist symmetric positive definite matrices $P, Q, R_1, R_2, S_1, S_2 \in \mathbb{R}^{n\times n}$, a matrix $S \in \mathbb{R}^{n\times n}$ and positive scalars $\lambda_i,\ i = \overline{1,7},\ \delta \geqslant 1$, such that the following matrix inequalities hold:*

$$\lambda_1R < P < \lambda_2R,\ Q < \lambda_3R,\ R_1 < \lambda_4R,\ R_2 < \lambda_5R,\ S_1 < \lambda_6I,\ S_2 < \lambda_7I, \tag{4}$$

$$\Omega_{h_1} = \begin{bmatrix} \Omega^{11} & \Omega^{12} & 0 & 0 & \Omega^{15} & 0 & \Omega^{17} & \Omega^{18} & \Omega^{19} & 0 \\ * & \Omega_{h_1}^{22} & \Omega_{h_1}^{23} & \Omega^{24} & 0 & 0 & 0 & 0 & 0 & \Omega_{h_1}^{2,10} \\ * & * & \Omega^{33} & \Omega_{h_1}^{34} & 0 & \Omega^{36} & 0 & 0 & 0 & \Omega_{h_1}^{3,10} \\ * & * & * & \Omega_{h_1}^{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Omega^{55} & 0 & \Omega^{57} & \Omega^{58} & \Omega^{59} & 0 \\ * & * & * & * & * & \Omega^{66} & \Omega^{67} & \Omega^{68} & \Omega^{69} & 0 \\ * & * & * & * & * & * & \Omega^{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Omega^{88} & 0 & 0 \\ * & * & * & * & * & * & * & * & \Omega^{99} & 0 \\ * & * & * & * & * & * & * & * & * & \Omega^{10,10} \end{bmatrix} < 0, \tag{5}$$

$$\Omega_{h_2}=\begin{bmatrix}\Omega^{11} & \Omega^{12} & 0 & 0 & \Omega^{15} & 0 & \Omega^{17} & \Omega^{18} & \Omega^{19} & 0\\ * & \Omega^{22}_{h_2} & \Omega^{23}_{h_2} & \Omega^{24} & 0 & 0 & 0 & 0 & 0 & 0\\ * & * & \Omega^{33} & \Omega^{34}_{h_2} & 0 & \Omega^{36} & 0 & 0 & 0 & \Omega^{3,10}_{h_2}\\ * & * & * & \Omega^{44}_{h_2} & 0 & 0 & 0 & 0 & 0 & \Omega^{4,10}_{h_2}\\ * & * & * & * & \Omega^{55} & 0 & \Omega^{57} & \Omega^{58} & \Omega^{59} & 0\\ * & * & * & * & * & \Omega^{66} & \Omega^{67} & \Omega^{68} & \Omega^{69} & 0\\ * & * & * & * & * & * & \Omega^{77} & 0 & 0 & 0\\ * & * & * & * & * & * & * & \Omega^{88} & 0 & 0\\ * & * & * & * & * & * & * & * & \Omega^{99} & 0\\ * & * & * & * & * & * & * & * & * & \Omega^{10,10}\end{bmatrix}<0, \tag{6}$$

$$\Lambda=[\Lambda^{ij}]_{7\times 7}<0. \tag{7}$$

Proof. Consider the following Lyapunov–Krasovskii functional:

$$V(k)=\sum_{i=1}^{4}V_i(k),$$

where

$$V_1(k)=x^{\mathrm{T}}(k)Px(k),$$
$$V_2(k)=\sum_{s=-h_2+1}^{-h_1+1}\sum_{t=k-1+s}^{k-1}\delta^{k-1-t}x^{\mathrm{T}}(t)Qx(t),$$
$$V_3(k)=\sum_{s=k-h_1}^{k-1}\delta^{k-1-s}x^{\mathrm{T}}(s)R_1x(s)+\sum_{s=k-h_2}^{k-h_1-1}\delta^{k-1-s}x^{\mathrm{T}}(s)R_2x(s),$$
$$V_4(k)=\sum_{s=-h_1+1}^{0}\sum_{t=k-1+s}^{k-1}h_1\delta^{k-1-t}y^{\mathrm{T}}(t)S_1y(t)+ +\sum_{s=-h_2+1}^{-h_1}\sum_{t=k-1+s}^{k-1}h_{12}\delta^{k-1-t}y^{\mathrm{T}}(t)S_2y(t).$$

By denoting $\eta(k):=[x^{\mathrm{T}}(k)\quad f^{\mathrm{T}}(x(k))\quad g^{\mathrm{T}}(x(k-h(k)))]^{\mathrm{T}}$, $\Gamma:=[A\quad W\quad W_1]$, we have the following estimates for the difference variation of $V_i(k), i=1,\dots,4$:

$$V_1(k+1)-\delta V_1(k)=\eta^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}P\Gamma\eta(k)-\delta x^{\mathrm{T}}(k)Px(k), \tag{8}$$

$$V_2(k+1)-\delta V_2(k)\leqslant(h_{12}+1)x^{\mathrm{T}}(k)Qx(k)-\delta^{h_1}x^{\mathrm{T}}(k-h(k))Qx(k-h(k)), \tag{9}$$

$$V_3(k+1)-\delta V_3(k)=x^{\mathrm{T}}(k)R_1x(k)+x^{\mathrm{T}}(k-h_1)[\delta^{h_1}(-R_1+R_2)]x(k-h_1) -\delta^{h_2}x^{\mathrm{T}}(k-h_2)R_2x(k-h_2), \tag{10}$$

$$V_4(k+1)-\delta V_4(k)\leqslant y^{\mathrm{T}}(k)[h_1^2S_1+h_{12}^2S_2]y(k)-h_1\delta\sum_{s=k-h_1}^{k-1}y^{\mathrm{T}}(s)S_1y(s) -h_{12}\delta^{h_1+1}\sum_{s=k-h_2}^{k-1-h_1}y^{\mathrm{T}}(s)S_2y(s). \tag{11}$$

By Proposition 2.1,

$$-h_1\delta\sum_{s=k-h_1}^{k-1}y^{\mathrm{T}}(s)S_1y(s)\leqslant-\delta[x(k)-x(k-h_1)]^{\mathrm{T}}S_1[x(k)-x(k-h_1)], \tag{12}$$

$$-h_{12}\delta^{h_1+1}\sum_{s=k-h_2}^{k-1-h_1}y^{\mathrm{T}}(s)S_2y(s)\leqslant-\delta^{h_1+1}\left(\frac{1}{(h(k)-h_1)/h_{12}}\zeta_1^{\mathrm{T}}S_2\zeta_1+\frac{1}{(h_2-h(k))/h_{12}}\zeta_2^{\mathrm{T}}S_2\zeta_2\right)$$
$$=-\delta^{h_1+1}\left(\frac{1}{\alpha}\zeta_1^{\mathrm{T}}S_2\zeta_1+\frac{1}{1-\alpha}\zeta_2^{\mathrm{T}}S_2\zeta_2\right)$$

$$= -\delta^{h_1+1}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\frac{1}{\alpha}S_2 & 0\\ 0 & \frac{1}{1-\alpha}S_2\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix},$$

where $\zeta_1 = x(k-h_1) - x(k-h(k))$, $\zeta_2 = x(k-h(k)) - x(k-h_2)$ and $\alpha = \frac{h(k)-h_1}{h_{12}}$.

Proposition 2.2 gives us

$$\begin{aligned}-h_{12}\delta^{h_1+1}\sum_{s=k-h_2}^{k-1-h_1} y^{\mathrm{T}}(s)S_2y(s) &\leqslant -\delta^{h_1+1}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}S_2+(1-\alpha)T_1 & S\\ S^{\mathrm{T}} & S_2+\alpha T_2\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}\\ &= -\delta^{h_1+1}[\zeta_1^{\mathrm{T}}(S_2+(1-\alpha)T_1)\zeta_1 + \zeta_1^{\mathrm{T}}S\zeta_2 + \zeta_2^{\mathrm{T}}S^{\mathrm{T}}\zeta_1\\ &\quad + \zeta_2^{\mathrm{T}}(S_2+\alpha T_2)\zeta_2], \qquad (13)\end{aligned}$$

where $T_1 = S_2 - SS_2^{-1}S^{\mathrm{T}}$ and $T_2 = S_2 - S^{\mathrm{T}}S_2^{-1}S$.

Substitute (12), (13) into (11) and combine with (8)-(10), we obtain

$$\begin{aligned}&V(k+1) - \delta V(k)\\ &\leqslant \eta^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}P\Gamma\eta(k) + x^{\mathrm{T}}(k)[-\delta P + (h_{12}+1)Q + R_1 - \delta S_1]x(k)\\ &\quad + x^{\mathrm{T}}(k)[2\delta S_1]x(k-h_1)\\ &\quad + x^{\mathrm{T}}(k-h_1)[\delta^{h_1}(-R_1+R_2) - \delta S_1 - \delta^{h_1+1}(S_2+(1-\alpha)T_1)]x(k-h_1)\\ &\quad + x^{\mathrm{T}}(k-h_1)[2\delta^{h_1+1}(S_2+(1-\alpha)T_1 - S)]x(k-h(k))\\ &\quad + x^{\mathrm{T}}(k-h_1)[2\delta^{h_1+1}S]x(k-h_2)\\ &\quad + x^{\mathrm{T}}(k-h(k))[-\delta^{h_1}Q - \delta^{h_1+1}(2S_2+(1-\alpha)T_1+\alpha T_2 - S - S^{\mathrm{T}})]x(k-h(k))\\ &\quad + x^{\mathrm{T}}(k-h(k))[2\delta^{h_1+1}(S_2+\alpha T_2 - S)]x(k-h_2)\\ &\quad + x^{\mathrm{T}}(k-h_2)[-\delta^{h_2}R_2 - \delta^{h_1+1}(S_2+\alpha T_2)]x(k-h_2)\\ &\quad + y^{\mathrm{T}}(k)[h_1^2S_1 + h_{12}^2S_2]y(k). \qquad (14)\end{aligned}$$

Besides, from (3), it is not hard to see that

$$\begin{aligned}0 &\leqslant f^{\mathrm{T}}(x(k))[-I]f(x(k)) + x^{\mathrm{T}}(k)[-2\overline{U}_2]f(x(k)) + x^{\mathrm{T}}(k)[-\overline{U}_1]x(k),\\ 0 &\leqslant g^{\mathrm{T}}(x(k-h(k)))[-I]g(x(k-h(k))) + x^{\mathrm{T}}(k-h(k))[-2\overline{V}_2]g(x(k-h(k)))\\ &\quad + x^{\mathrm{T}}(k-h(k))[-\overline{V}_1]x(k-h(k)). \qquad (15)\end{aligned}$$

Furthermore, by setting

$$\xi(k) = [x^{\mathrm{T}}(k) \quad x^{\mathrm{T}}(k-h_1) \quad x^{\mathrm{T}}(k-h(k)) \quad x^{\mathrm{T}}(k-h_2) \quad f^{\mathrm{T}}(x(k)) \quad g^{\mathrm{T}}(x(k-h(k)))]^{\mathrm{T}}$$

$$\Upsilon = \begin{bmatrix} PA & 0 & 0 & 0 & PW & PW_1\\ h_1^2S_1(A-I) & 0 & 0 & 0 & h_1^2S_1W & h_1^2S_1W_1\\ h_{12}^2S_2(A-I) & 0 & 0 & 0 & h_{12}^2S_2W & h_{12}^2S_2W_1\end{bmatrix},$$

we can rewrite

$$\eta^{\mathrm{T}}(k)\Gamma^{\mathrm{T}}P\Gamma\eta(k)+y^{\mathrm{T}}(k)[h_1^2S_1+h_{12}^2S_2]y(k)=\xi^{\mathrm{T}}(k)\Upsilon^{\mathrm{T}}\begin{bmatrix}P & 0 & 0\\ 0 & h_1^2S_1 & 0\\ 0 & 0 & h_{12}^2S_2\end{bmatrix}^{-1}\Upsilon\xi(k). \quad (16)$$

Consequently, combining (14), (15) and (16) gives

$$V(k+1)-\delta V(k)\leqslant \xi^{\mathrm{T}}(k)\left(\Phi_{\mathrm{h(k)}}+\Upsilon^{\mathrm{T}}\begin{bmatrix}\mathrm{P} & 0 & 0\\ 0 & \mathrm{h}_1^2\mathrm{S}_1 & 0\\ 0 & 0 & \mathrm{h}_{12}^2\mathrm{S}_2\end{bmatrix}^{-1}\Upsilon\right)\xi(k), \quad (17)$$

where

$$\Phi_{h(k)}:=\begin{bmatrix}\Omega^{11} & \Omega^{12} & 0 & 0 & \Omega^{15} & 0\\ * & \Phi_{h(k)}^{22} & \Phi_{h(k)}^{23} & \Omega^{24} & 0 & 0\\ * & * & \Phi_{h(k)}^{33} & \Phi_{h(k)}^{34} & 0 & \Omega^{36}\\ * & * & * & \Phi_{h(k)}^{44} & 0 & 0\\ * & * & * & * & \Omega^{55} & 0\\ * & * & * & * & * & \Omega^{66}\end{bmatrix},$$

with

$$\Phi_{h(k)}^{22}=\delta^{h_1}[-R_1+R_2]-\delta S_1-\delta^{h_1+1}[S_2+(1-\alpha)T_1],$$

$$\Phi_{h(k)}^{23}=\delta^{h_1+1}[S_2+(1-\alpha)T_1-S],$$

$$\Phi_{h(k)}^{33}=-\delta^{h_1}Q-\delta^{h_1+1}[2S_2+(1-\alpha)T_1+\alpha T_2-S-S^{\mathrm{T}}]-\overline{V}_1,$$

$$\Phi_{h(k)}^{34}=\delta^{h_1+1}[S_2+\alpha T_2-S],\qquad \Phi_{h(k)}^{44}=-\delta^{h_2}R_2-\delta^{h_1+1}[S_2+\alpha T_2].$$

Next, by using Proposition 2.3, it can be deduced that

$$\Phi_{h(k)}+\Upsilon^{\mathrm{T}}\begin{bmatrix}P & 0 & 0\\ 0 & h_1^2S_1 & 0\\ 0 & 0 & h_{12}^2S_2\end{bmatrix}^{-1}\Upsilon<0 \quad \Leftrightarrow \quad \Psi_{h(k)}<0,$$

where

$$\Psi_{h(k)}:=\begin{bmatrix}\Omega^{11} & \Omega^{12} & 0 & 0 & \Omega^{15} & 0 & \Omega^{17} & \Omega^{18} & \Omega^{19}\\ * & \Phi_{h(k)}^{22} & \Phi_{h(k)}^{23} & \Omega^{24} & 0 & 0 & 0 & 0 & 0\\ * & * & \Phi_{h(k)}^{33} & \Phi_{h(k)}^{34} & 0 & \Omega^{36} & 0 & 0 & 0\\ * & * & * & \Phi_{h(k)}^{44} & 0 & 0 & 0 & 0 & 0\\ * & * & * & * & \Omega^{55} & 0 & \Omega^{57} & \Omega^{58} & \Omega^{59}\\ * & * & * & * & * & \Omega^{66} & \Omega^{67} & \Omega^{68} & \Omega^{69}\\ * & * & * & * & * & * & \Omega^{77} & 0 & 0\\ * & * & * & * & * & * & * & \Omega^{88} & 0\\ * & * & * & * & * & * & * & * & \Omega^{99}\end{bmatrix}.$$

It follows from convex combination technique that $\Psi_{h(k)}<0$ if the following two inequalities

$$\Psi_{h_1}<0 \quad \text{and} \quad \Psi_{h_2}<0$$

hold. By Proposition 2.3 again, the above two inequalities hold if the inequalities stated in (5) and (6) hold. This, together with (17), shows that

$$V(k+1) - \delta V(k) \leqslant 0 \quad \forall k \in \mathbb{Z}_+.$$

By iteration, it infers that

$$V(k) \leqslant \delta^k V(0) \leqslant \delta^N V(0) \quad \forall k = 1,2,\ldots,N. \tag{18}$$

From assumption (4) and $x(k) = \varphi(k)\ \forall k \in \{-h_2, -h_2+1, \ldots, 0\}$, we have the following estimate

$$\begin{aligned} V(0) \leqslant & \left[\lambda_2 + \lambda_3 \delta^{h_2-1} \tfrac{h_2(h_2+1)-h_1(h_1-1)}{2} + \lambda_4 \delta^{h_1-1} h_1 + \lambda_5 \delta^{h_2-1}(h_2-h_1)\right] c_1 \\ & + \left[\lambda_6 \delta^{h_1-1} h_1 \tfrac{h_1(h_1+1)}{2} + \lambda_7 \delta^{h_2-1} h_{12} \tfrac{h_2(h_2+1)-h_1(h_1+1)}{2}\right] \tau \ =: \sigma \end{aligned} \tag{19}$$

From (18) and (19), we get

$$V(k) < \delta^N \sigma. \tag{20}$$

On the other hand, from (4) it follows that

$$V(k) \geqslant x^{\mathrm{T}}(k) P x(k) \geqslant \lambda_1 x^{\mathrm{T}}(k) R x(k) \quad \forall k \in \mathbb{Z}_+. \tag{21}$$

Note that by Proposition 2.3, the inequality (7) is equivalent to

$$\begin{aligned} -c_2\delta\lambda_1 + c_1\delta^{N+1}\lambda_2 + \rho_1\lambda_3 + c_1 h_1 \delta^{N+h_1}\lambda_4 + c_1 h_{12}\delta^{N+h_2}\lambda_5 \\ + \tfrac{1}{2}\tau h_1^2(h_1+1)\delta^{N+h_1}\lambda_6 + \rho_2\lambda_7 < 0, \end{aligned}$$

or

$$-c_2\delta\lambda_1 + \delta^{N+1}\sigma < 0. \tag{22}$$

Accordingly, from (20)-(22), we find that:

$$x^{\mathrm{T}}(k) R x(k) < \tfrac{1}{\delta\lambda_1}[\delta^{N+1}\sigma] < c_2 \quad \forall k = 1,2,\ldots,N.$$

This implies that system (1) is finite-time stable with respect to (c_1, c_2, R, N). □

Remark 3.1. Conditions (4)-(7) are in the form of matrix inequalities and (5)-(7) will become linear matrix inequalities (LMIs) when we fix the parameter δ and thus they can be tested for feasibility easily with MATLAB software.

Remark 3.2. In [4] we used reciprocally convex combination technique and in this work we applied an extension of that technique. To see that the criteria proposed in Theorem 3.1 are better than those given in [4], let's consider the following example.

Example 3.1. Consider the system (1), where

$$A = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad W = \begin{bmatrix} -0.025 & 0.025 \\ 0.02 & 0.035 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.05 & 0.025 \\ -0.05 & 0.025 \end{bmatrix},$$

$$U_1 = 0.5I_2, \quad U_2 = 0.9I_2, \quad V_1 = 0.1I_2, \quad V_2 = 0.5I_2,$$

$$R = \begin{bmatrix} 1.25 & 0.25 \\ 0.25 & 1.35 \end{bmatrix}, \quad h(k) = 2 + 25\sin^2 \frac{k\pi}{2}, k \in Z_+.$$

Since $\overline{U}_1 = 0.45I_2$, $\overline{U}_2 = -0.7I_2 \neq 0$, $\overline{V}_1 = 0.05I_2$, $\overline{V}_2 = -0.3I_2 \neq 0$, i.e., the neuron activation functions in this case are sector-bounded, Corollary 3.1 of [4] is not applicable.

For given $h_1 = 2$, $h_2 = 27$, $N = 90$, $\tau = 1$, $c_1 = 1$, and $c_2 = 9$, the LMIs (4)-(7) are feasible with $\delta = 1.0001$ and

$$P = \begin{bmatrix} 18.5370 & 7.0746 \\ 7.0746 & 23.7728 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.0690 & 0.0329 \\ 0.0329 & 0.0622 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.1346 & 0.0779 \\ 0.0779 & 0.4112 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0.0612 & 0.0077 \\ 0.0077 & 0.0732 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0.2434 & 0.0072 \\ 0.0072 & 0.2225 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.0071 & -0.0000 \\ -0.0000 & 0.0069 \end{bmatrix},$$

$$S = \begin{bmatrix} -0.0071 & 0.0001 \\ 0.0000 & -0.0066 \end{bmatrix}, \quad \lambda_1 = 12.9389, \quad \lambda_2 = 18.7143, \quad \lambda_3 = 0.0648,$$

$$\lambda_4 = 0.4694, \quad \lambda_5 = 0.0674, \quad \lambda_6 = 0.2975, \quad \lambda_7 = 0.0072.$$

For this reason, by Theorem 3.1, the system is finite-time stable w.r.t. $(1, 9, R, 90)$.

4. CONCLUSION

In this paper, we address the finite-time stability for a general class of discrete-time neural networks subjected to interval-like time-varying delay and sector-bounded neuron activation functions. By using an extended reciprocally convex matrix inequality, we have achieved a refined delay-dependent sufficient condition that can be quickly programmed and computed by the LMI Toolbox in MATLAB.

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VỀ MỘT TIÊU CHUẨN ỔN ĐỊNH TRONG THỜI GIAN HỮU HẠN CHO LỚP HỆ NƠ-RON RỜI RẠC CÓ TRỄ VỚI CÁC HÀM KÍCH HOẠT NƠ-RON BỊ CHẶN KIỂU HÌNH QUẠT

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TÓM TẮT

Bài báo nghiên cứu tính ổn định trong thời gian hữu hạn của lớp hệ nơ-ron rời rạc với các hàm kích hoạt nơ-ron bị chặn kiểu hình quạt và độ trễ biến thiên theo thời gian dạng khoảng. Cách tiếp cận lồi nghịch đảo mở rộng được sử dụng để thiết lập một điều kiện đủ phụ thuộc độ trễ nhằm đảm bảo tính ổn định trong thời gian hữu hạn của lớp hệ này. Bài báo cũng chứa một ví dụ bằng số nhằm minh họa tính hiệu quả của tiêu chuẩn đã đề xuất.

Từ khóa: Hệ nơ-ron rời rạc, tính ổn định trong thời gian hữu hạn, bất đẳng thức ma trận tuyến tính, độ trễ biến thiên theo thời gian.

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